Conditional expectations on compact quantum groups and new examples of quantum hypergroups.

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Abstract

We propose a construction of quantum hypergroups using conditional expectations on compact quantum groups. Using this construction, we describe several series of non-trivial finite-dimensional quantum hypergroups via conditional expectations of Delsart type on non-trivial Kac algebras obtained by twisting of the finite groups.

0. Introduction

The compact quantum hypergroup was introduced in [3] as a structure that simultaniosly generalies usual hypergroups ([1], [2]), compact quantum groups ([16]) and bialgebras of biinvariant functions associated with quantum Gelfand pairs ([9], [4], [13], [15]). A compact quantum hypergroup is a unital C^*-algebra equipped with a coassociative completely positive coproduct that preserves the unit element and satisfies some additional axioms. In [3] a theory of corepresentations was studied and there was established an analog of the Peter-Weyl theory for compact quantum hypergroups.

Howewer, as we know, the only *concrete* nontrivial example of the compact quantum hypergroup was constucted in [11]. The main purpose of this paper is to build a sufficient number of such examples. We propose a general construction of quantum hypergroups using conditional expectations on compact quantum groups and, using this construction, we carry, to the quantum case, the Delsart construction of hypergroups (recall that if G is a locally compact group and Γ is a compact group of automorphisms of G with a Haar measure μ_γ, then the algebra of continuous Γ-invariant functions on G equipped with the coproduct

$$(\Delta f)(x, y) = \int_\Gamma f(x\gamma(y))d\mu_\gamma,$$

is a hypergroup).

Using the Delsart construction we build several series of non-trivial finite-dimensional quantum hypergroups from non-trivial Kac algebras obtained in [14], [10] by twisting the classical series of finite groups.

The paper is organized as follows. Section 1 contains neessary preliminaries on compact quantum hypergroups and on twisting the finite groups. In Section 2 we describe the construction of compact quantum hypergroups using conditional expectations on quantum hypergroups (and, in particular, on compact quantum groups). In the commutative case, we compare this construction with orbital morphisms [7]. In Section 3 we show that this construction includes the double coset construction in quantum groups [4], [13], [15] and an analogue of the Delsart construction for quantum hypergroup. We also give here sufficient conditions for an authomorphism of the initial group (or a certain automorphism constructed from it) to be an automorphism of the corresponding twisted Kac algebra. Section 4 contains five series of nontrivial examples of finite-dimensional Delsart quantum hypergroups associated with twistings of the classical series of finite groups.

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1. Preliminaries

1.1. Let $(A, \cdot, 1, *)$ be a separable unital C^*-algebra. We denote by $A \otimes A$ the injective or projective C^*-tensor square of A. We will call $(A, \Delta, \epsilon, \star)$ a *hypergroup structure* on the C^*-algebra $(A, \cdot, 1, *)$ if:

(HS$_1$) $(A, \Delta, \epsilon, \star)$ is a $\star$-coalgebra with a counit ϵ, i.e. $\Delta : A \to A \otimes A$ and $\epsilon : A \to \mathbf{C}$ are linear mappings, $\star : A \to A$ is an antilinear mapping such that

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \tag{1.1}$$

$$(\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id, \tag{1.2}$$

$$\Delta \circ \star = \Sigma \circ (\star \otimes \star) \circ \Delta, \tag{1.3}$$

$$\star \circ \star = id, \tag{1.4}$$

where $\Sigma : A \otimes A \to A \otimes A$ is the flip $\Sigma(a_1 \otimes a_2) = a_2 \otimes a_1$;

(HS$_2$) the mapping $\Delta : A \to A \otimes A$ is positive, i.e. it maps the cone of positive elements of A into the cone of positive elements of $A \otimes A$;

(HS$_3$) the following identities hold

$$(a \cdot b)^\star = a^\star \cdot b^\star, \qquad \Delta \circ * = (* \otimes *) \circ \Delta, \tag{1.5}$$

$$\epsilon(a \cdot b) = \epsilon(a)\epsilon(b), \qquad \Delta(1) = 1 \otimes 1, \tag{1.6}$$

$$\star \circ * = * \circ \star. \tag{1.7}$$

1.2. Let $(A, \Delta, \epsilon, \star)$ be a hypergroup structure on a C^*-algebra A. By A' we denote the set of all continuous linear functionals on the C^*-algebra A. For $\xi, \eta \in A'$ we define a product $\cdot$ and an involution $^+$ by

$$(\xi \cdot \eta)(a) = (\xi \otimes \eta)\delta(a),$$
$$\xi^+(a) = \overline{\xi(a^\star)}, \tag{1.8}$$

$a \in A$, with the norm given by

$$\|\xi\| = \sup_{\|a\|=1} |\xi(a)|. \tag{1.9}$$

Then A' is a unital Banach $*$-algebra.

A state $\mu \in A'$ is called a *Haar measure* (with respect to the hypergroup structure) if

$$(\mu \otimes id) \circ \Delta(a) = (id \otimes \mu) \circ \Delta(a) = \mu(a)1 \tag{1.10}$$

for all $a \in A$.

Let $(A, \Delta, \epsilon, \star)$ be a hypergroup structure on the C^*-algebra A. An element $a \in A$ is called *positive definite* if

$$\xi \cdot \xi^+(a) \geq 0 \tag{1.11}$$

for all $\xi \in A'$.

1.3. Theorem ([3]). *Let $(A, \Delta, \epsilon, \star)$ be a hypergroup structure on a C^*-algebra A. Suppose that the linear space spanned by positive definite elements is dense in A. Then there exists a Haar measure μ, it is unique, and $\mu^+ = \mu$.*

1.4. Suppose that $(A, \Delta, \epsilon, \star)$ is a hypergroup structure on a C^*-algebra $(A, \cdot, 1, *)$ and μ is the corresponding Haar measure. We call $\mathcal{A} = (A, \Delta, \epsilon, \star, \sigma_t, \mu)$ a *compact quantum hypergroup* if

(QH_1) the mapping Δ is completely positive and the linear span of positive definite elements is dense in A;

(QH_2) σ_t, $t \in \mathbb{R}$, is a continuous one-parameter group of automorphisms of A such that

(a) there exist dense subalgebras $A_0 \subset A$ and $\tilde{A}_0 \subset A \otimes A$ such that the one-parameter groups σ_t and $\sigma_t \otimes$ id, id $\otimes \sigma_t$ can be extended to complex one-parameter groups σ_z and $\sigma_z \otimes$ id, id $\otimes \sigma_z$, $z \in \mathbb{C}$, of automorphisms of the algebras A_0 and $\tilde{A}_0$ respectively;

(b) A_0 is invariant with respect to $*$ and $\star$, and $\Delta(A_0) \subset \tilde{A}_0$;

(c) the following relations hold on A_0 for all $z \in \mathbb{C}$:

$$\Delta \circ \sigma_z = (\sigma_z \otimes \sigma_z) \circ \Delta, \tag{1.12}$$
$$\mu(\sigma_z(a)) = \mu(a); \tag{1.13}$$

(d) there exists $z_0 \in \mathbb{C}$ such that the Haar measure μ satisfies the following strong invariance condition for all $a, b \in A_0$:

$$(id \otimes \mu)\big[\big((* \circ \sigma_{z_0} \circ \star \otimes id) \circ \Delta(a)\big) \cdot (1 \otimes b)\big] = (id \otimes \mu)\big((1 \otimes a) \cdot \Delta(b)\big);$$
(1.14)

(QH_3) the Haar measure μ is faithful on A_0.

It will be convenient to denote

$$\kappa = * \circ \sigma_{z_0} \circ \star \tag{1.15}$$

and call it an *antipode*. Then for all $a, b \in A_0$,

$$\kappa(ab) = \kappa(b)\kappa(a), \qquad \delta \circ \kappa = \Pi \circ (\kappa \otimes \kappa) \circ \delta,$$
$$\nu \circ \kappa = \nu, \qquad \kappa(1) = 1, \qquad \epsilon \circ \kappa = \epsilon. \tag{1.16}$$

Note that κ is invertible with $\kappa^{-1} = \star \circ \sigma_{-z_0} \circ *$.

With such a notation, relation (1.14) becomes

$$(id \otimes \mu)\big((\kappa \otimes id) \circ \Delta(a) \cdot (1 \otimes b)\big) = (id \otimes \mu)\big((1 \otimes a) \cdot \Delta(b)\big). \tag{1.17}$$

1.5. Compact quantum groups are quantum hypergroups. In detail, let $\mathcal{A} = (A, \Delta, \epsilon, \kappa)$ be a compact matrix pseudogroup with A_0 being the involutive subalgebra generated by matrix elements of the fundamental corepresentation [16]. We will use the following notations

$$\xi \cdot a = (id \otimes \xi) \circ \Delta(a), \qquad a \cdot \xi = (\xi \otimes id) \circ \Delta(a), \qquad \xi \cdot \eta = (\xi \otimes \eta) \circ \Delta \tag{1.18}$$

for $\xi, \eta \in A'$ and $a \in A$. It readily follows from (1.18) that

$$\xi . (\eta . a) = (\xi \cdot \eta) . a, \qquad (a . \eta) . \xi = a . (\eta \cdot \xi). \tag{1.19}$$

Let $U^\alpha = (u_{ij}^\alpha)_{i,j=1}^{d_\alpha}$ be an irreducible unitary corepresentation of A. Then there exists a unique, up to a positive constant, positive definite matrix $M^\alpha = (m_{ij}^\alpha)_{i,j=1}^{d_\alpha}$ such that

$$M^\alpha \cdot U^\alpha = S^2(U^\alpha) \cdot M^\alpha, \tag{1.20}$$

where $\cdot$ here denotes the usual matrix multiplication [16].

For each $z \in \mathbb{C}$, we denote by $m_{ij}^{\alpha(z)}$ the matrix elements of the matrix $(M^\alpha)^z$. It is known that there exists a one-parameter family of homomorphisms $f_z : A_0 \to \mathbb{C}$, $z \in \mathbb{C}$, where, as before, A_0 denotes the $*$-subalgebra generated by matrix elements of the fundamental corepresentation. These homomorphisms are defined by

$$f_z(u_{ij}^\alpha) = m_{ij}^{\alpha(z)} \tag{1.21}$$

and possess the following properties [16]:

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(F_1): $f_z(1) = 1$ for all $z \in \mathbb{C}$;

(F_2): $f_z \cdot f_{z'} = f_{z+z'}$ and $f_0 = \epsilon$;

(F_3): $f_z(\kappa(a)) = f_{-z}(a)$;

(F_4): $f_z(a^*) = \overline{f_{-\overline{z}}(a)}$;

(F_5): $\kappa^2(a) = f_{-1} \cdot a \cdot f_1$;

(F_6): $\mu(a \cdot b) = \mu(b \cdot (f_1 \cdot a \cdot f_1))$, where μ is the Haar measure on the compact quantum group A.

It was shown in [3] that $(A, \Delta, \epsilon, \star, \sigma_t)$ is quantum hypergroup, where the mapping $\star : A_0 \to A_0$ is defined by

$$a^\star = f_{-1/2} \cdot S(a)^* \cdot f_{1/2}, \tag{1.22}$$

the action of the group σ_t is defined by

$$\sigma_t(a) = f_{it} \cdot a \cdot f_{-it} \tag{1.23}$$

and $z_0 = -\frac{1}{2}i$.

1.6. Here we briefly discuss the case of usual compact hypergroups, or equivalently, normal hypercomplex systems with basis unity ([1]). Let $\mathcal{A}$ be a commutative compact quantum hypergroup. Let Q denote the spectrum of the commutative C^*-algebra A. Each element $\xi \in Q$ defines a linear operator on A given by

$$R_\xi = (id \otimes \xi) \circ \Delta. \tag{1.24}$$

The operators $R_\xi, \xi \in Q$, constitute a family of generalized translation operators on $A \cong C(Q)$. For $\xi \in Q$ and $a \in A_0$, we define

$$\xi^\dagger(a) = \overline{\xi(a^\dagger)}, \tag{1.25}$$

where

$$a^\dagger = \kappa(a)^*. \tag{1.26}$$

From the definition of $a^\dagger$, it immediately follows that $\xi^\dagger$ is a homomorphism $A_0 \to \mathbb{C}$ and, hence, continuous. Being extended by continuity to A, it becomes a point in Q. In [3] it was established that Q is a basis of a normal hypercomplex system $L_1(Q, \mu)$ with a basis unit ϵ.

1.7. Preliminary on twisting of Kac algebras. We recall the construction of a twisting of Kac algebras following [5], [14]. For our purposes, it suffices to deal only with the case of finite dimensional Kac algebras.

Let $\mathcal{A} = (A, \Delta, \varepsilon, \kappa)$ be a finite dimensional Kac algebra. A *2-cocycle* of A is a unitary Ω in $A \otimes A$ such that

$$(\Omega \otimes 1)(\Delta \otimes \mathrm{id})(\Omega) = (1 \otimes \Omega)(\mathrm{id} \otimes \Delta)(\Omega).$$

A *2-pseudo-cocycle* of A is a unitary Ω in $A \otimes A$ such that

$$\partial_2 \Omega = (\mathrm{id} \otimes \Delta)(\Omega^*)(1 \otimes \Omega^*)(\Omega \otimes 1)(\Delta \otimes \mathrm{id})(\Omega)$$

belongs to $(\Delta \otimes \mathrm{id})\Delta(A)'$ ([5], 2.3). Let us put, for all x in A :

$$\Delta_\Omega(x) = \Omega\Delta(x)\Omega^*.$$

Then Δ_Ω is coassociative iff Ω is a 2-pseudo-cocycle of A; and we shall say that Δ_Ω is *twisted* from Δ by Ω.

For a unitary $u \in A$, let us put $\Omega^u = (u^* \otimes u^*)\Omega\Delta(u)$. Then Ω^u is a 2(-pseudo)-cocycle iff Ω is. Let $\Sigma(x \otimes y) = y \otimes x$ be the flip in $A \otimes A$. We shall say that a 2(-pseudo)-cocycle Ω is *pseudo-co-involutive* (resp., *co-involutive*) if $\Omega^u(\Sigma(\kappa \otimes \kappa)(\Omega^*))^* \in \Delta(A)'$ (resp.,$\Sigma(\kappa \otimes \kappa)(\Omega^*) = \Omega^u$). We shall say that Ω is *strongly co-involutive*, if $\Sigma(\kappa \otimes \kappa)(\Omega^*) = \Omega$. It should be noted that each *counital* 2-cocycle Ω of a finite dimensional Kac algebra (i. e. such that $(\varepsilon \otimes \mathrm{id})\Omega = (\mathrm{id} \otimes \varepsilon)\Omega = 1$) is coinvolutive with $u = m(\mathrm{id} \otimes \kappa)\Omega = \kappa(u)$ and $\varepsilon(u) = 1$ (here m denotes the multiplication in the algebra A). If Ω is a (pseudo)-co-involutive 2(-pseudo)-cocycle of a Kac algebra A, then the coalgebra (A, Δ_Ω) possesses a coinvolution of the form $\kappa_\Omega(x) = u\kappa(x)u^*$. If Ω is a strongly coinvolutive 2-cocycle of a Kac algebra A, then κ is a coinvolution of the coalgebra (A, Δ_Ω).

Let G be a finite group and H be its Abelian subgroup. Let $\mathbb{C}(G)$ be the C^*-algebra generated by the left regular representation λ of G. Then $\mathbb{C}(G)$ has a standart structure of cocommutative Kac algebra $(\mathbb{C}(G), \Delta, \kappa, \mu)$, where $\Delta(\lambda(g)) = \lambda(g) \otimes \lambda(g)$, $\kappa(\lambda(g)) = \lambda(g^{-1})$, $\mu(\lambda(f)) = f(e)$, $g \in G$ and f is a continuous function on G. Denote by $\hat{H}$ the dual group of H. Then there exists a family of projections $P_{\hat{h}}$ generating an Abelian subalgebra $\mathbb{C}(H)$ in $\mathbb{C}(G)$ such that

$$P_{\hat{h}}P_{\hat{g}} = \delta_{\hat{h},\hat{g}}P_{\hat{h}}, \ \lambda(h) = \sum_{\hat{h} \in \hat{H}} <\hat{h}, h> P_{\hat{h}}, \ P_{\hat{h}} = \frac{1}{||H||} \sum_{h \in H} <\hat{h}, h> \lambda(h),$$

$$\Delta(P_{\hat{h}}) = \sum_{\hat{g} \in \hat{H}} P_{\hat{g}} \otimes P_{\hat{g}^{-1}\hat{h}}, \ \kappa(P_{\hat{h}}) = P_{\hat{h}^{-1}},$$

where $g, h \in H$, $\hat{g}, \hat{h} \in \hat{H}$, $||H||$ is the order of H. Using these idempotents, one can write the following formulae for Ω and u:

$$\Omega = \sum_{x,y \in \hat{H}} \omega(x, y)(P_x \otimes P_y), \ u = m(i \otimes \kappa)\Omega = \sum_{x \in \hat{H}} \omega(x, x^{-1})P_x,$$

where $\omega : \hat{H} \times \hat{H} \to \mathbb{T}$ is a 2-(pseudo)-cocycle. If ω is a 2-cocycle, then so is Ω, but if ω is only a 2-pseudo-cocycle on $\hat{H}$, one should verify that Ω is a 2-pseudo-cocycle on $(\mathbb{C}(G), \Delta, \kappa)$. If it is, and it is at least pseudo-coinvolutive, then we have already a new finite dimensional coinvolutive coalgebra. If it has a counit, then it is a Kac algebra. For this, one should choose ω to be counital, $\omega(e, x) = \omega(x, e) = 1 \ \forall x \in H$, which gives the counitality of Ω.

2. Conditional expectations on quantum hypergroups

Let $(A, \Delta, \epsilon, \star, \sigma_t, \mu)$ be a compact quantum hypergroup and κ be the antipod defined by (1.15). Let B be a unital C^*-subalgebra of A and $P : A \to B$ be a corresponding μ-invariant conditional expectation [12] (i. e. $\mu \circ P = \mu$). The following Theorem 2.1 states that, under some conditions, we can define, on the algebra B, a new comultiplication $\tilde{\Delta}$ such that $(B, \tilde{\Delta}, \epsilon, \star, \sigma_t, \mu)$ is a compact quantum hypergroup.

2.1. Theorem. *Let $\mathcal{A}$ be a compact quantum hypergroup, B an unital C^*-subalgebra of a C^*-algebra A and $P : A \to B$ a μ-invariant conditional expectation. Then, let us put for all $x \in B$,*

$$\tilde{\Delta}(x) = (P \otimes P)\Delta(x). \tag{2.1}$$

Suppose that the following conditions hold:

(1) $(P \otimes P)\Delta(x) = (P \otimes P)\Delta(P(x))$ (or, equivalently, $\ker P$ is a coideal in A),

(2) $P \circ \star = \star \circ P$,

(3) the dense subalgebras $A_0 \subset A$ and $\tilde{A}_0 \subset A \otimes A$ are invariant under P and $P \otimes P$, respectively, and $P \circ \sigma_z = \sigma_z \circ P$ for all $z \in \mathbb{C}$,

(4) the mapping ϵ is a counit of the coalgebra $(B, \tilde{\Delta})$, i. e., the relation

$$(\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id$$

holds.

Then $(B, \tilde{\Delta}, \varepsilon, \star, \sigma_t, \mu)$ is a compact quantum hypergroup.

Proof. First we prove that $(B, \tilde{\Delta}, \epsilon, \star)$ is a hypergroup structure. Using the first condition of the theorem, we have for all $x \in B$

$$
\begin{aligned}
(\tilde{\Delta} \otimes \text{ id})\tilde{\Delta}(x) &= (P \otimes P \otimes \text{ id})(\Delta \otimes \text{ id})(P \otimes P)\Delta(x) \\
&= (\text{ id} \otimes \text{ id} \otimes P)((P \otimes P) \circ \Delta \circ P \otimes \text{ id})\Delta(x) \\
&= (P \otimes P \otimes P)(\Delta \otimes \text{ id})\Delta(x) \\
&= (P \otimes P \otimes P)(\text{ id} \otimes \Delta)\Delta(x) = (\text{ id} \otimes \tilde{\Delta})\tilde{\Delta}(x),
\end{aligned}
$$

i.e., the mapping $\tilde{\Delta} : B \to B \otimes B$ is coassociative. Relation (1.2) follows from Condition 4 and relations (1.3), (1.4) easily follow from the second condition of the theorem. The mapping $\tilde{\Delta}$ is completely positive since P is completely positive and Δ is a homomorphism. Finally, identities (1.5)-(1.7) are obvious.

Let A', B' be the spaces of continuous linear functionals on A, B, respectively, equipped with the multiplications

$$\langle \xi \cdot \eta, a \rangle = \langle \xi \otimes \eta, \Delta(a) \rangle \qquad (\xi, \eta \in A'; a \in A)$$

and

$$\langle \xi' \cdot \eta', b \rangle = \left\langle \xi' \otimes \eta', \tilde{\Delta}(b) \right\rangle \qquad (\xi', \eta' \in B'; b \in B).$$

In order to show that the linear span of positive definite elements is dense in B, let us define a mapping $P' : B' \to A'$ as follows:

$$\langle P'\xi, \; a \rangle = \langle \xi, \; Pa \rangle, \tag{2.2}$$

where $a \in A, \xi \in B'$. If $a \in A$ is a positive definite element then $Pa \in B$ is a positive definite element (with respect to $\tilde{\Delta}$). Indeed, for all $\xi \in B'$ we have

$$
\begin{aligned}
\langle \xi \cdot \xi^+, Pa \rangle &= \langle \xi \otimes \overline{\xi}, (id \otimes \star)(P \otimes P)\Delta(Pa) \rangle \\
&= \langle \xi \otimes \overline{\xi}, (P \otimes P)(id \otimes \star)\Delta(a) \rangle \\
&= \langle P'\xi \otimes \overline{P'\xi}, (id \otimes \star)\Delta(a) \rangle \geq 0,
\end{aligned}
$$

because $\langle \eta \cdot \eta^+, a \rangle = \langle \eta \otimes \overline{\eta}, (id \otimes \star)\Delta(a) \rangle \geq 0$ for all $\eta \in A'$. So the axiom (QH_1) holds for the hypergroup structure $(B, \tilde{\Delta}, \epsilon, \star)$. Since the linear span of $\{Pa \in B \mid a \text{ is positive definite in } A\}$ is dense in B, we get the result.

Define $B_0 = PA_0$ and $\tilde{B}_0 = P \otimes P(\tilde{A}_0)$. Conditions 2 and 3 of the theorem imply that B_0 and $\tilde{B}_0$ are dense subalgebras of B and $B \otimes B$, respectively, such that axioms (QH_2)(a) - (QH_2)(c) hold.

Let us check the remaining axioms. The faithful state μ is a Haar measure of the hypergroup structure $(B, \tilde{\Delta}, \epsilon, \star)$, since P is μ-invariant. Let us show that μ is strongly invariant with respect to $\tilde{\Delta}$, i.e. relation (1.17) holds for the hypergroup structure $(B, \tilde{\Delta}, \epsilon, \star)$. It follows from Conditions 2 and 3 of the theorem, that P commutes with the antipode κ defined by (1.15). Using this fact, the fact that a Haar measure in a quantum hypergroup $\mathcal{A}$ is strongly invariant, and since P is μ-invariant, we have for all $a, b \in B$ that

$$
\begin{aligned}
&(id \otimes \mu)[((\kappa \otimes id) \circ \tilde{\Delta}(a)) \cdot (1 \otimes b)] \\
={}& (id \otimes \mu)[((P \otimes P) \circ (\kappa \otimes id) \circ \Delta(a)) \cdot (1 \otimes b)] \\
={}& (id \otimes \mu) \circ (P \otimes P)[((\kappa \otimes id) \circ \Delta(a)) \cdot (1 \otimes b)] \\
={}& (P \otimes \mu)[((\kappa \otimes id) \circ \Delta(a)) \cdot (1 \otimes b)] \\
={}& (P \otimes \mu)[(1 \otimes a) \cdot \Delta(b))] = (id \otimes \mu)[(P \otimes P)((1 \otimes a) \cdot \Delta(b)] \\
={}& (id \otimes \mu)[(1 \otimes a) \cdot \tilde{\Delta}(b)].
\end{aligned}
$$

□

2.2. Remark. We can simplify the hypothesis of Theorem 2.1 in case when $\mathcal{A}$ is a compact quantum group. Indeed, let $\mathcal{A} = (A, \Delta, \epsilon, \kappa)$ be a compact matrix pseudogroup with A_0 being the $*$-subalgebra generated by matrix elements of the fundamental corepresentation. Let μ be its Haar measure and $P : A \to B$ be a μ-invariant conditional expectation that maps to a unital C^*-subalgebra B of A. Let us define a new comultiplication $\tilde{\Delta}$ on B in accordance with (2.1). Then the statement of Theorem 2.1 remains true if we replace the hypothesis 2 and 3 with the following condition:

(A) A_0 is invariant with respect to P and $\kappa \circ P = P \circ \kappa$.

Proof. In this case we have that $\tilde{A}_0 \subset A_0 \otimes A_0$ and we only need to prove that P commutes with $\star$ and σ_t. Using property (F_5) of f_z we have for all $a \in A_0$,
$$P(f_{-1} \cdot a \cdot f_1) = P(\kappa^2(a)) = \kappa^2(P(a)) = f_{-1} \cdot Pa \cdot f_1,$$
whence $P(f_{-n} \cdot a \cdot f_n) = f_{-n} \cdot Pa \cdot f_n$ for all $n \in \mathbb{N}$. Since $f_z(a)$ is an entire function of exponential growth on the right half-plane for all $a \in A_0$, the vector-valued function
$$f_{-z} \cdot u_{kl}^\alpha \cdot f_z = \sum_{r,s} f_z(\kappa(u_{kr}^\alpha)) u_{rs}^\alpha f_z(u_{sl}^\alpha)$$

is entire of exponential growth on the right half-plane in the weak sence, and we obtain (cf. [16], Lemma 5.5) that $P(f_{-z} \cdot a \cdot f_z) = (f_{-z} \cdot Pa \cdot f_z)$ for all $z \in \mathbb{C}$ and $a \in A_0$, whence it follows from (1.23) that P commutes with σ_z, $z \in \mathbb{C}$. Also, for $a \in A_0$ by using (1.22), we have $P(a^\star) = P(f_{-1/2} \cdot \kappa(a)^* \cdot f_{1/2}) = f_{-1/2} \cdot P((\kappa(a))^*) \cdot f_{1/2} = f_{-1/2} \cdot (\kappa(Pa))^* \cdot f_{1/2} = (Pa)^\star.$ □

In order to clarify Condition 4 of Theorem 2.1, we establish sufficient conditions for the mapping ϵ be a counit of the coalgebra $(B, \tilde{\Delta})$. The following Remark 2.3 is quite transparent, but its condition does not hold in some examples. Therefore, we establish Proposition 2.4 with less restrictive conditions on ϵ in the case where $\mathcal{A}$ is a finite dimensional Kac algebra (finite dimensional quantum groups are, in fact, Kac algebras [16]).

2.3. Remark. Let a μ-invariant conditional expectation $P : A \to B$ on a compact quantum hypergroup $\mathcal{A}$ satisfy Condition 1 of Theorem 2.1. If the mapping $\epsilon : A \to \mathbb{C}$ satisfies the relation $\epsilon \circ P = \epsilon$, then the mapping $\epsilon : B \to \mathbb{C}$ is a counit of the coalgebra $(B, \tilde{\Delta})$, i. e. (1.2) holds.

2.4. Proposition. *Let a μ-invariant conditional expectation $P : A \to B$ on a finite dimensional Kac algebra $\mathcal{A}$ satisfy Condition 1 of Theorem 2.1. Denote by p_ϵ the one-dimensional central projection in A such that the relation $ap_\epsilon = \epsilon(a)p_\epsilon$ holds for all $a \in A$. Let the following conditions hold for all $b \in B$*

$$(P \otimes P)[(p_\epsilon \otimes 1) \cdot \Delta(b)] \quad = \quad (P \otimes P)[(P(p_\epsilon) \otimes 1) \cdot \Delta(b)] \qquad (2.3)$$
$$(P \otimes P)[(1 \otimes p_\epsilon) \cdot \Delta(b)] \quad = \quad (P \otimes P)[(1 \otimes P(p_\epsilon)) \cdot \Delta(b)]. \qquad (2.4)$$

Then the mapping $\epsilon : B \to \mathbb{C}$ is a counit of the finite dimensional coalgebra $(B, \tilde{\Delta})$.

Proof. First we prove that relations (1.2) hold if and only if, for all $b \in B$, we have

$$
\begin{aligned}
(P(p_\epsilon) \otimes 1)\tilde{\Delta}(b) &= P(p_\epsilon) \otimes b, & (2.5)\\
(1 \otimes P(p_\epsilon))\tilde{\Delta}(b) &= b \otimes P(p_\epsilon). & (2.6)
\end{aligned}
$$

Indeed, from (2.5) with the usual tensor notation $\tilde{\Delta}(b) = b_{(1)} \otimes b_{(2)}$, we have

$$
\begin{aligned}
P(p_\epsilon) \otimes b &= (P(p_\epsilon) \otimes 1)\tilde{\Delta}(b) = P(p_\epsilon b_{(1)}) \otimes b_{(2)}\\
&= \epsilon(b_{(1)})P(p_\epsilon) \otimes b_{(2)} = P(p_\epsilon) \otimes \epsilon(b_{(1)})b_{(2)}\\
&= P(p_\epsilon) \otimes (\epsilon \otimes id)\tilde{\Delta}(b).
\end{aligned}
$$

Similarly, we have $b \otimes P(p_\epsilon) = (id \otimes \epsilon)\tilde{\Delta}(b) \otimes P(p_\epsilon)$. The converse statement follows by similar arguments.

Now by using (2.3) we get (2.5):

$$
\begin{aligned}
(P(p_\epsilon) \otimes 1)\tilde{\Delta}(b) &= (P \otimes P)[(P(p_\epsilon) \otimes 1) \cdot \Delta(b)]\\
&= (P \otimes P)[(p_\epsilon \otimes 1) \cdot \Delta(b)] = (P \otimes P)[(p_\epsilon \otimes b]\\
&= P(p_\epsilon) \otimes b.
\end{aligned}
$$

Similarly, (2.6) follows from (2.4). $\qquad \square$

2.5. Lemma. *Let $\mathcal{A}$ be a compact quantum hypergroup and all hypothesis of Theorem 2.1 hold for a conditional expectation $P : A \to B$. Let $P' : B' \to A'$ be defined by (2.2). Then P' is a $*$-homomorphism.*

Proof. For any $\eta, \eta_1, \eta_2 \in B'$ and $a \in A$, we have

$$
\begin{aligned}
\langle P'(\eta_1 \cdot \eta_2), a \rangle &= \langle \eta_1 \cdot \eta_2, Pa \rangle = \left\langle \eta_1 \otimes \eta_2, \tilde{\Delta}(Pa) \right\rangle\\
&= \langle \eta_1 \otimes \eta_2, (P \otimes P)\Delta(Pa) \rangle = \langle \eta_1 \otimes \eta_2, (P \otimes P)\Delta(a) \rangle\\
&= \langle P'(\eta_1) \otimes P'(\eta_2), \Delta(a) \rangle = \langle P'(\eta_1) \cdot P'(\eta_2), a \rangle \,;\\
\langle P'(\eta^+), a \rangle &= \overline{\langle \eta, P(a)^\star \rangle} = \overline{\langle \eta, P(a^\star) \rangle} = \langle (P'\eta)^+, a \rangle \,.
\end{aligned}
$$

$\qquad \square$

2.6. In what follows we discuss conditional expectations on usual hypergroups and a construction of orbital morphisms [7]. For simplicity, we consider only the case of compact hypergroups. Let Q be a compact DJS-hypergroup with involution $Q \ni p \mapsto p^\dagger \in Q$, comultiplication $\Delta : C(Q) \to C(Q) \otimes C(Q)$, neutral element (counit) e and Haar measure μ, and let Y be a compact Hausdorff space. Let ϕ be an open continuous mapping from Q onto Y (orbital mapping). The closed sets $\phi^{-1}(y)$, $y \in Y$, are called ϕ-*orbits*. Let B be a C^*-subalgebra of $C(Q)$ consisting of functions constant on ϕ-orbits. Obviously, a mapping $\phi : C(Y) \to B$ defined by $(\phi f)(x) = f(\phi(x))$ is an isomorphism of

the C^*-algebras. Denote by $\phi_* : M(Q) \to M(Y)$ the corresponding mapping of Radon measures, $\langle \phi_*(\mu), f \rangle = \langle \mu, f \circ \phi \rangle$, $\mu \in M(Q)$, $f \in C(Y)$. The measure $\mu \in M(Q)$ is called ϕ-consistent, if $\phi_*(\mu * \nu) = 0 = \phi_*(\nu * \mu)$ whenever $\phi_*(\nu) = 0$. The following proposition clarifies the concept of ϕ-consistency.

2.7. Proposition. *Let $\phi : Q \to Y$ be an orbital mapping and denote $\tilde{e} = \phi(e)$. Suppose that the following conditions are satisfied:*

(a) *$\phi^{-1}(\tilde{e}) = \{e\}$,*

(b) *if A is ϕ-orbit then so is $A^\dagger$,*

(c) *for any $y \in Y$ there exists a probability measure $q_y \in M(Q)$ such that $\operatorname{supp} q_y \subset \phi^{-1}(y)$,*

(d) *each measure q_y is ϕ-consistent.*

Define the linear mapping $P : C(Q) \to B$ as follows:

$$(Pf)(x) = \langle q_{\phi(x)}, f \rangle. \tag{2.7}$$

Then P is a conditional expectation and satisfies all hypothesis of Theorem 2.1. Conversely, if conditions (a)–(c) hold and the linear mapping P defined by (2.7) is a conditional expectation from $C(Q)$ to B satisfying all hypothesis of Theorem 2.1, then each q_y is ϕ-consistent.

Proof. By virtue of (b) we can define an involutive homeomorphism $\dagger$: $Y \to Y$ as follows: if $y = \phi(x)$, then $y^\dagger = \phi(x^\dagger)$. Theorem 13.5A in [7] states that there exists a unique convolution $*$ in $M(Y)$ such that Y is a hypergroup and ϕ is an orbital morphism, i. e.,

(i) $\delta_y * \delta_z = \phi_*(q_y * q_z)$ for any $y, z \in Y$,

(ii) $q_{y^\dagger} = (q_y)^\dagger$,

(iii) $\operatorname{supp} q_y = \phi^{-1}(y)$ and

$$m = \int_Q q_{\phi(x)} m(dx), \tag{2.8}$$

where m is a Haar measure on the hypergroup Q.

Thus we can define a mapping $\phi^* : M(Y) \to M(Q)$ by setting $\phi^*(\delta_y) = q_y$ and

$$\phi^*(\nu) = \int_Y q_z\ \nu(dz),$$

for $y \in Y$ and $\nu \in M(Y)$. In virtue of Lemma 13.6A in [7], the orbital morphism ϕ is consistent, i. e. the mapping ϕ^* is a $*$-homomorphism. It also follows from the proof of Theorem 13.5A cited above that the mapping $Y \ni y \mapsto q_y \in M(q)$

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is continuous in the weak topology. Thus P is well-defined and is, indeed, a conditional expectation. Let $f \in \ker P$. Then $\langle q_z, f \rangle = 0$ for all $z \in Y$, and for $x_1, x_2 \in Q$ we have

$$
\begin{aligned}
((P \otimes P)\Delta f)(x_1, x_2) &= \langle q_{\phi(x_1)} \otimes q_{\phi(x_2)}, \Delta f \rangle \\
&= \langle q_{\phi(x_1)} * q_{\phi(x_2)}, f \rangle = \langle \phi^*(\delta_{\phi(x_1)}) * \phi^*(\delta_{\phi(x_2)}), f \rangle \\
&= \langle \phi^*(\delta_{\phi(x_1)} * \delta_{\phi(x_2)}), f \rangle = \int_Y \langle q_z, f \rangle \ (\delta_{\phi(x_1)} * \delta_{\phi(x_2)})(dz) \\
&= \int_Y (Pf)(\phi(x)) \ (\delta_{\phi(x_1)} * \delta_{\phi(x_2)})(dz) = 0,
\end{aligned}
$$

where $x \in \phi^{-1}(z)$. Hence, $\ker P$ is a coideal. In virtue of (iii), P is m-invariant and the fourth condition of Theorem 2.1 follows from (a). At last, the equality $P \circ \dagger = \dagger \circ P$ follows from (b) (note that, for usual hypergroups, $f^\star(x) = \overline{f(x^\dagger)}$ and $\sigma_z = id$).

Let us prove the converse statement. Define the convolution in $M(Y)$ in accordance with (i). Since B is isomorfic to $C(Y)$, it follows from Theorem 2.1 that Y is a DJS-hypergroup. To prove the result, we need to show that ϕ is a consistent orbital morphism. Then the result follows from Theorem 13.6B in [7]. Indeed, (ii) follows from (b) and equality (2.8) follows from the fact that P is m-invariant. Let us show that $\operatorname{supp} q_y = \phi^{-1}(y)$. Suppose that $x_0 \in \phi^{-1}(y)$ and $x_0 \notin \operatorname{supp} q_y$. Then there exists an open neighborhood O_{x_0} of x_0 such that $O_{x_0} \cap \operatorname{supp} q_y = \varnothing$. Let $f \in C(Q)$ be a positive function such that $f(x_0) = 0$ and $f(x) = 1$ for $x \in Q \backslash O_{x_0}$. Since $Pf \in C(Q)$ and $(Pf)(x_0) = \langle q_y, f \rangle = 1$, one can find, for an arbitrary $0 < \varepsilon < 1/2$, an open neighborhood $V \subset O_{x_0}$ of x_0 such that $f(x) < \epsilon$ and $(Pf)(x) > 1 - \varepsilon$ for $x \in V$. Denote by i_V the indicator of V. Since

$$
\langle q_{\phi(x)}, i_V f \rangle = \int_V f(t) q_{\phi(x)}(dt) \geq (1 - \varepsilon) q_{\phi(x)}(V),
$$

we have, by using (2.8),

$$
\begin{aligned}
\varepsilon m(V) &\geq \int_V f(x) m(dx) = \int_Q \langle q_{\phi(x)}, i_V f \rangle \, m(dx) \\
&\geq (1 - \varepsilon) \int_Q q_{\phi(x)}(V) m(dx) = (1 - \varepsilon) m(V).
\end{aligned}
$$

Since m is positive on open sets, we have that $\operatorname{supp} q_y = \phi^{-1}(y)$. The fact that the orbital morphism ϕ is consistent follows from Lemma 2.5. $\qquad\square$

2.8. Remark. The first condition of Theorem 2.1 is not necessary for coassociativity of $\tilde{\Delta}$. Indeed, the decomposition of $\mathbb{Z}_6 = \{0\} \cup \{1, 2\} \cup \{3\} \cup \{4, 5\}$ on ϕ-orbits is a DJS-hypergroup, but $\tilde{\Delta}(\delta_1 - \delta_2) \neq 0$, although $\delta_1 - \delta_2 \in \ker P$.

3. Quantum double cosets and quantum Delsart hyper-groups

In this section we use Theorem 2.1 to introduce double cosets of quantum groups [4], [13] and an analogue of Delsart construction for a quantum hypergroup.

3.1. Quantum double cosets. Let $\mathcal{A}_i = (A_i, \delta_i, \epsilon_i, \kappa_i)$, $i = 1, 2$, be two compact matrix pseudogroups and let $\pi : A_1 \to A_2$ be a Hopf C^*-algebra epimorphism, i. e. π is a C^*-algebra epimorphism satisfying $(\pi \otimes \pi) \circ \Delta_1 = \Delta_2 \circ \pi$, $\epsilon_2 \circ \pi = \epsilon_1$ and also $\pi(A_{10}) \subset A_{20}$ with $\pi \circ \kappa_1 = \kappa_2 \circ \pi$, where A_{i0} is the $*$-subalgebra of A_i generated by matrix elements of the fundamental corepresentation. It was established in [3] that $\pi \circ \star = \star \circ \pi$, where $\star$ is defined by (1.22), on each A_i, $i = 1, 2$. Let μ_i be a Haar measure of $\mathcal{A}_i$, $i = 1, 2$. The algebra A_1 possesses a structure of a left (right) comodule with respect to the coactions $\Delta_l(a) = (\pi \otimes \text{id})\Delta(a)$ (resp., $\Delta_r(a) = (\text{id} \otimes \pi)\Delta(a)$). Define

$$A_1/A_2 = \{a \in A_1 : (id \otimes \pi) \circ \delta(a) = a \otimes 1\},$$

$$A_2\backslash A_1 = \{a \in A_1 : (\pi \otimes id) \circ \delta(a) = 1 \otimes a\}, \tag{3.1}$$

$$A_2\backslash A_1/A_2 = A_2\backslash A_1 \bigcap A_1/A_2.$$

It is immediate that A_1/A_2, $A_2\backslash A_1$, $A_2\backslash A_1/A_2$ are involutive algebras with the unit 1. Denote by A^{inv} the C^*-algebra completion of $A_2\backslash A_1/A_2$.

Define $P = \pi^l \circ \pi^r$, where $\pi^l = (\mu_2 \circ \pi \otimes \text{id}) \circ \Delta_1$, $\pi^r = (\text{id} \otimes \mu_2 \circ \pi) \circ \Delta_1$ are commuting projections on A_1. Then $P : A_1 \to A^{\text{inv}}$ is a conditional expectation on A_1 satisfying hypotheses of Theorem 2.1. Hence $\mathcal{B} = (A^{\text{inv}}, \tilde{\Delta}, \varepsilon_1, \star, \sigma_t, \mu_1)$ is a compact quantum hypergroup, where σ_t and $\star$ are defined by (1.22), (1.23). Also, for all $b \in A^{\text{inv}}$, the following formula for the comultiplication $\tilde{\Delta}$ holds:

$$\tilde{\Delta}(b) = (\text{ id } \otimes \mu_2 \circ \pi \otimes \text{ id })(\Delta_1 \otimes \text{ id})\Delta_1(b). \tag{3.2}$$

Indeed, the projections π^l, π^r satisfy the equalities

$$(\text{ id } \otimes \pi^r) \circ \Delta = \Delta \circ \pi^r, \quad (\pi^l \otimes \text{ id }) \circ \Delta = \Delta \circ \pi^l. \tag{3.3}$$

By using these equalities and straightforward calculation, one can check that $\ker P$ is a coideal. Formula (3.2) easily follows from (3.3):

$$
\begin{aligned}
(P \otimes P)\Delta_1(b) &= (\pi^r \otimes \pi^l) \circ \Delta_1(\pi^l \circ \pi^r(b)) = (\pi^r \otimes \pi^l) \circ \Delta_1(b) \\
&= (\pi^r \otimes \text{ id} \otimes \text{ id}) \circ (\text{ id} \otimes \mu_2 \circ \pi \otimes \text{ id}) \circ (\text{ id} \otimes \Delta_1) \circ \Delta_1(b) \\
&= (\pi^r \otimes \text{ id} \otimes \text{ id}) \circ (\text{ id} \otimes \mu_2 \circ \pi \otimes \text{ id}) \circ (\Delta_1 \otimes \text{ id}) \circ \Delta_1(b) \\
&= (\pi^r \otimes \text{ id}) \circ \Delta_1(b) = (\text{ id} \otimes \mu_2 \circ \pi \otimes \text{ id}) \circ (\Delta_1 \otimes \text{ id}) \circ \Delta_1(b).
\end{aligned}
$$

It is obvious that P is μ_1-invariant:

$$
\begin{aligned}
\mu \circ P(a) &= (\mu_1 \otimes \mu_2 \circ \pi) \circ \Delta_1 \circ (\mu_2 \circ \pi \otimes \text{ id}) \circ \Delta_1(a) \\
&= (\mu_2 \circ \pi \otimes \mu_1 \otimes \mu_2 \circ \pi) \circ (\text{ id} \otimes \Delta_1) \circ \Delta_1(a) \\
&= \mu_1(a)(\mu_2 \circ \pi \otimes \mu_2 \circ \pi)(1 \otimes 1) = \mu_1(a).
\end{aligned}
$$

It is also obvious that P commutes with κ_1 and that condition (A) of Remark 2.2 holds. In virtue of Remark 2.2 we only need to examine Condition 4 of Theorem 2.1. To prove Condition 4, we have by using (3.2) that

$$
\begin{aligned}
(\varepsilon_1 \otimes \text{ id }) \circ \tilde{\Delta}(a) &= (\varepsilon_1 \otimes \text{ id }) \circ (\text{ id} \otimes \mu_2 \circ \pi \otimes \text{ id}) \circ (\Delta_1 \otimes \text{ id }) \circ \Delta_1(a) \\
&= (\mu_2 \circ \pi \otimes \text{ id}) \circ (\varepsilon_1 \otimes \text{ id} \otimes \text{ id}) \circ (\Delta_1 \otimes \text{ id}) \circ \Delta_1(b) \\
&= (\mu_2 \circ \pi \otimes \text{ id}) \Delta_1(b) = \pi^l(b) = b
\end{aligned}
$$

for all $b \in A_2 \backslash A_1 / A_2$. Similarly, $(\text{ id} \otimes \varepsilon_1) \circ \tilde{\Delta}(b) = b$. $\qquad \square$

3.2. Remark. If quantum groups $\mathcal{A}_1$, $\mathcal{A}_2$ are finite dimensional, then *condition 4 of Theorem 2.1 follows from Proposition 2.4.* Indeed, it is obvious that π^l (resp. π^r) is a conditional expectation from A_1 to $A_2 \backslash A_1$ (resp. to A_1/A_2). Conditions (2.3), (2.4) of Proposition 2.4 then follow from the following relations

$$
\begin{aligned}
(P \otimes P)((a \otimes 1)(\tilde{\Delta}(b))) &= (P \otimes P)((Pa \otimes 1)(\tilde{\Delta}(b))) \\
(P \otimes P)((1 \otimes a)(\tilde{\Delta}(b))) &= (P \otimes P)((1 \otimes Pa)(\tilde{\Delta}(b))),
\end{aligned}
$$

which hold for all $b \in A_2 \backslash A_1 / A_2$, $a \in A_1$. The last relations follows by direct computations from the fact that π^l and π^r are conditional expectations.

3.3. Delsart hypergroups. Let $\mathcal{A} = (A, \delta, \epsilon, \kappa)$ be a compact matrix pseudogroup and let Γ be a compact group of Hopf C^*-algebra automorphisms of $\mathcal{A}$, i. e. each $\gamma \in \Gamma$ is a C^*-algebra automorphism satisfying $(\gamma \otimes \gamma) \circ \Delta = \Delta \circ \gamma$, $\epsilon \circ \gamma = \epsilon$ and also $\gamma(A_0) \subset A_0$ with $\gamma \circ \kappa = \kappa \circ \gamma$, where A_0 is the $*$-subalgebra of A generated by matrix elements of the fundamental corepresentation. Let ν be a Haar measure of Γ such that $\nu(\Gamma) = 1$. In what follows, we denote integration with respect to ν by $d\gamma$. Denote by $B = \{b \in A | \gamma(b) = b\}$ the fixed point algebra for the Γ-action. Define a mapping $P : A \to B$ by $Pa = \int_\Gamma \gamma(a)$. Then *$P$ is a conditional expectation on A satisfying hypothesis of Theorem 2.1. Hence, $\mathcal{B} = (B, \tilde{\Delta}, \varepsilon, \star, \sigma_t, \mu)$ is a compact quantum hypergroup (Delsart hypergroup), where σ_t and $\star$ are defined by (1.22), (1.23). Also, for all $b \in B$, the following formula for the comultiplication $\tilde{\Delta}$ holds:*

$$
\tilde{\Delta}(b) = (\text{ id } \otimes P)(\Delta(b). \tag{3.4}
$$

Indeed, it is obvious that P is a conditional expectation. Since, for any $\gamma \in \Gamma$ and $a \in A$,

$$
(\mu \circ \gamma)(a)1 = (\text{ id} \otimes \mu)\Delta(\gamma(a)) = \gamma((\text{ id} \otimes \mu \circ \gamma)\Delta(a)),
$$

we have that $\mu \circ \gamma$ is a Haar measure. Since a normalized Haar measure is unique, we have that $\mu \circ \gamma = \mu$, whence it follows that P is μ-invariant. Since $\varepsilon \circ \gamma = \varepsilon$ we get that $\varepsilon \circ P = \varepsilon$ and condition 4 of Theorem 2.1 follows from Remark 2.3. The facts that $\ker P$ is a coideal and that P commutes with κ are obvious. Thus the statement follows from Remark 2.2. $\qquad \square$

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3.4. In order to construct nontrivial examples of finite dimensional quantum Delsart hypergroups, we need to know about automorphisms of nontrivial Kac algebras. A number of examples of nontrivial finite dimensional Kac algebras are constructed in [14], [10] as twistings of the Kac algebras of finite groups. It is natural to expect that automorphisms of twisted Kac algebras are related with automorphisms of the corresponding finite groups. In what follows, we state some results in this direction.

Let $(\mathbb{C}(G), \Delta_\Omega, \varepsilon, \kappa_\Omega)$ be a Kac algebra obtained by twisting from the co-commutative Kac algebra $(\mathbb{C}(G), \Delta, \varepsilon, \kappa)$ of a finite group G with respect to an abelian subgroup H of G. Let α be an automorphism of G. Define $\alpha(\lambda(g)) = \lambda(\alpha(g))$. Then $\alpha : \mathbb{C}(G) \rightarrow \mathbb{C}(G)$ is an automorphism of the Kac algebra $(\mathbb{C}(G), \Delta, \varepsilon, \kappa)$. Denote by Ad Ω the automorphism of $\mathbb{C}(G) \otimes \mathbb{C}(G)$ given by Ad $\Omega(x) = \Omega x \Omega^*$, where $x \in \mathbb{C}(G) \otimes \mathbb{C}(G)$.

The next two propositions give sufficient conditions for α (or a certain automorphism constructed from α) to be an automorphism of the twisted Kac algebra $(\mathbb{C}(G), \Delta_\Omega, \varepsilon, \kappa_\Omega)$.

3.5. Proposition. *Let Ω be a (pseudo)-coinvolutive 2-(pseudo)-cocycle of $\mathbb{C}(G)$. If $\alpha \otimes \alpha$ commutes with Ad Ω, then α is an automorphism of the twisted Kac algebra $(\mathbb{C}(G), \Delta_\Omega, \varepsilon, \kappa_\Omega)$. In particular, it is sufficient that $\alpha \restriction H = $ id.*

Proof. The first statement is obvious. The last statement follows from the fact that $\Omega \in C(H) \otimes \mathbb{C}(H)$. $\qquad\qquad\square$

3.6. Proposition. *Let Ω be a (pseudo)-coinvolutive 2-(pseudo)-cocycle of $\mathbb{C}(G)$ and $u = m(\text{ id} \otimes \kappa)\Omega$ be the corresponding unitary in $\mathbb{C}(G)$. Denote by γ the automorphism of this C^*-algebra defined by $\gamma(x) = u\alpha(x)u^*$ for $x \in \mathbb{C}(G)$. Suppose that the following conditions hold:*

1. *$\alpha(u) = u^*$,*

2. *the element $(\alpha^{-1} \otimes \alpha^{-1})(\Omega^u)\Omega^*$ belongs to the commutant of $\Delta(\mathbb{C}(G))$,*

3. *$\varepsilon(u) = 1$.*

Then γ is an automorphism of the coalgebra $(\mathbb{C}(G), \Delta_\omega, \varepsilon)$, i. e. $(\gamma \otimes \gamma) \circ \Delta_\Omega = \Delta_\Omega \circ \gamma$, $\varepsilon \circ \gamma = \varepsilon$. If $\kappa(u) = u$, then γ is an automorphism of the Kac algebra $(\mathbb{C}(G), \Delta_\omega, \varepsilon, \kappa_\Omega)$. The last condition is always true when Ω is a counital 2-cocycle, but if Ω is only a 2-pseudococycle, then one should verify that γ commutes with κ_Ω.

Proof. It follows from the first condition of the proposition that $\gamma = $ Ad $u \circ \alpha = \alpha \circ$ Ad u^*. Following condition 2 one can find an element Z in commutant

of $\Delta(\mathbb{C}(G))$ such that $(\alpha^{-1} \otimes \alpha^{-1})(\Omega^u) = \Omega Z$. Then, for any $x \in \mathbb{C}(G)$, we have

$$
\begin{aligned}
\Delta_\Omega(\gamma(x)) &= \Omega\Delta(u\alpha(x)u^*)\Omega^* = \Omega\Delta(u)((\alpha \otimes \alpha)\Delta(x))\Delta(u^*)\Omega^* \\
&= (\alpha \otimes \alpha)\left((\alpha^{-1} \otimes \alpha^{-1})(\Omega\Delta(u))\Delta(x)(\alpha^{-1} \otimes \alpha^{-1})(\Delta(u^*)\Omega^*)\right) \\
&= (\gamma \otimes \gamma)\left((u \otimes u)(\alpha^{-1} \otimes \alpha^{-1})(\Omega\Delta(u))\Delta(x)(\alpha^{-1} \otimes \alpha^{-1})(\Delta(u^*)\Omega^*)(u^* \otimes u^*)\right) \\
&= (\gamma \otimes \gamma)\left((\alpha^{-1} \otimes \alpha^{-1})((u^* \otimes u^*)\Omega\Delta(u))\Delta(x)(\alpha^{-1} \otimes \alpha^{-1})(\Delta(u^*)\Omega^*(u \otimes u))\right) \\
&= (\gamma \otimes \gamma)\left((\alpha^{-1} \otimes \alpha^{-1})(\Omega^u)\Delta(x)(\alpha^{-1} \otimes \alpha^{-1})(\Omega^u)^*\right) \\
&= (\gamma \otimes \gamma)(\Omega\Delta(x)\Omega^*) \\
&= (\gamma \otimes \gamma)(\Delta_\Omega(x)).
\end{aligned}
$$

It follows from condition 3 of the proposition that $\varepsilon \circ \gamma = \varepsilon$. To prove the last statement, let us suppose that $\kappa(u) = u$. Then, for any $x \in \mathbb{C}(G)$, we have $\gamma(\kappa_\Omega(x)) = u\alpha(u\kappa(x)u^*)u^* = \alpha(\kappa(x)) = \kappa(\alpha(x))$. On the other hand, $\kappa_\Omega(\gamma(x)) = u\kappa(u\alpha(x)u^*)u^* = u\kappa(u^*)\kappa(\alpha(x))\kappa(u)u^* = \kappa(\alpha(x))$. $\square$

3.7. Remark. By using the action of an automorphism α on the group $\hat{H}$ which is dual to the Abelian subgroup H, $< \alpha(\hat{h}), h > = < \hat{h}, \alpha^{-1}(h) >$, one can rewrite condition 1 of Proposition 3.7 in terms of the 2-(pseudo)-cocycle ω on $\hat{H} \times \hat{H}$,

$$
\omega(\alpha(\hat{x}), \alpha(\hat{x}^{-1})) = \overline{\omega(\hat{x}, \hat{x}^{-1})} \quad (\hat{x} \in \hat{H}).
$$

Condition 2 of Proposition 3.7 follows from the relation $(\alpha^{-1} \otimes \alpha^{-1})(\Omega^u) = \Omega$ which can also be rewritten in terms of the 2-(pseudo)-cocycle ω,

$$
\omega(\hat{x}, \hat{x}^{-1})\omega(\hat{y}, \hat{y}^{-1})\omega(\alpha(\hat{x}), \alpha(\hat{y})) = \omega(\hat{x}, \hat{y})\omega(\hat{x}\hat{y}, (\hat{x}\hat{y})^{-1}) \quad (\hat{x}, \hat{y} \in \hat{H}).
$$

One can also rewrite other equalities that contain Ω in such manner, for example, the relation $(\alpha^{-1} \otimes \alpha^{-1})\Omega = \Omega$ is sufficient for the statement of Proposition 3.6 to hold,

$$
\omega(\alpha(\hat{x}), \alpha(\hat{y})) = \overline{\omega(\hat{x}, \hat{y})}. \tag{3.5}
$$

4. Examples

4.1. Quantum hypergroups associated with a twisting of the quasi-quaternionic group. Let $G = Q_n$ $(n = 2, 3, \dots)$ be the quasiquaternionic group generated by two elements, a of order $2n$ and b of order 4 such that $b^2 = a^n$ and $bab^{-1} = a^{-1}$. The group $Q_n = \{a^k, ba^k, k = 0, 1, \dots, 2n - 1\}$ and the group algebra $A = \mathbb{C}(G)$ is isomorphic to

$$
\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \underbrace{M_2(\mathbb{C}) \oplus \cdots \oplus M_2(\mathbb{C})}_{n-1}.
$$

Let e_1, e_2, e_3, e_4, e_{11}^j, e_{12}^j, e_{21}^j, e_{22}^j $(j = 1, \dots, n-1)$, be the matrix units of this algebra. We can now write the left regular representation λ of G [6],

$$
\lambda(a^k) = e_1 + e_2 + (-1)^k(e_3 + e_4) + \sum_j (\varepsilon_n^{jk} e_{11}^j + \varepsilon_n^{-jk} e_{22}^j),
$$

$$\lambda(a^k b) = e_1 - e_2 + (-1)^k(e_3 - e_4) + \sum_j(\varepsilon_n^{j(k-n)}e_{12}^j + \varepsilon_n^{-jk}e_{21}^j),$$

for even n, and

$$\lambda(a^k b) = e_1 - e_2 + i(-1)^k(e_3 - e_4) + \sum_j(\varepsilon_n^{j(k-n)}e_{12}^j + \varepsilon_n^{-jk}e_{21}^j),$$

for odd n, where $\varepsilon_n = e^{i\pi/n}$.

Consider the subgroup $H = \mathbb{Z}_4 = \{e, b, b^2, b^3\}$. Since the dual group $\hat{H}$ is isomorphic to H, following [14], one can compute the orthogonal projections $P_{\hat{h}}$ $(\hat{h} \in \hat{H})$,

$$
\begin{aligned}
P_{\hat{e}} &= e_1 + e_3 + q_1, \ P_{\hat{b}} = p_1, \\
P_{\hat{b}^2} &= e_2 + e_4 + q_2, \ P_{\hat{b}^3} = p_2, \ for \ even \ n, \\
P_{\hat{e}} &= e_1 + q_1, \ P_{\hat{b}} = e_4 + p_1, \\
P_{\hat{b}^2} &= e_2 + q_2, \ P_{\hat{b}^3} = e_3 + p_2, \ for \ odd \ n,
\end{aligned}
$$

where p_1, p_2, q_1, q_2 are the orthogonal projections defined by :

$$
\begin{aligned}
p_1 &= 1/2\sideset{}{'}\sum_j(e_{11}^j - ie_{12}^j + ie_{21}^j + e_{22}^j), \\
p_2 &= 1/2\sideset{}{'}\sum_j(e_{11}^j + ie_{12}^j - ie_{21}^j + e_{22}^j), \\
q_1 &= 1/2\sideset{}{''}\sum_j(e_{11}^j + e_{12}^j + e_{21}^j + e_{22}^j), \\
q_2 &= 1/2\sideset{}{''}\sum_j(e_{11}^j - e_{12}^j - e_{21}^j + e_{22}^j),
\end{aligned}
$$

where $\sum'$ (resp., $\sum''$) means that the corresponding index in the summation takes only odd (resp., even) values. One can see that e_1 is the projection given by the co-unit of A.

The Abelian subalgebra generated by $\lambda(e)$, $\lambda(b), \lambda(b^2)$, $\lambda(b^3)$ is also generated by the mutually orthogonal orthogonal projections $P_{\hat{e}}$, $P_{\hat{b}}$, $P_{\hat{b}^2}$, $P_{\hat{b}^3}$. The projections $P_{\hat{e}} + P_{\hat{b}^2}$ and $P_{\hat{b}} + P_{\hat{b}^3}$ are central.

The unitary $\Omega \in A \otimes A$ is obtained by the lifting construction from the pseudo-cocycle ω on $\hat{H} \times \hat{H}$ such that, for all u, h in $\hat{H}$,

$$\omega(\hat{e}, u) = \omega(u, u) = 1, \overline{\omega(h, u)} = \omega(u, h)$$

and $\omega(\hat{b}, \hat{b}^2) = \omega(\hat{b}^2, \hat{b}^3) = \omega(\hat{b}^3, \hat{b}) = i$. In [14] it was established that

$$
\begin{aligned}
\Omega &= P_{\hat{e}} \otimes I + P_{\hat{b}} \otimes (P_{\hat{e}} + P_{\hat{b}} + i(P_{\hat{b}^2} - P_{\hat{b}^3})) + P_{\hat{b}^2} \otimes (P_{\hat{e}} + P_{\hat{b}^2} \\
&+ i(P_{\hat{b}^3} - P_{\hat{b}})) + P_{\hat{b}^3} \otimes (P_{\hat{e}} + P_{\hat{b}^3} + +i(P_{\hat{b}} - P_{\hat{b}^2}))
\end{aligned}
$$

is a pseudo-coinvolutive 2-pseudo-cocycle with respect to the unitary

$$u = P_{\hat{e}} + P_{\hat{b}^2} + i(P_{\hat{b}^3} - P_{\hat{b}})$$

such that $\mathcal{A}_\Omega = (A, \Delta_\Omega, \varepsilon, \kappa_\Omega, \mu)$ is a nonsymmetric Kac algebra (i. e. $\Sigma \circ \Delta_\Omega \neq \Delta_\Omega$).

Define an authomorphism α of order 2 on $G = Q_n$: $\alpha(a) = a$, $\alpha(b) = b^3$. Let us verify that α satisfyes the hypothesis of Proposition 3.7. Indeed, $\alpha(P_{\hat{b}}) = P_{\hat{b}^3}$ and, hence, $\alpha(u) = u^*$. Let $Z = (\alpha \otimes \alpha)(\Omega^u)\Omega^*$. Then it is not hard to compute that

$$Z = (P_{\hat{e}} + P_{\hat{b}^2}) \otimes I + (P_{\hat{b}} + P_{\hat{b}^3}) \otimes ((P_{\hat{e}} + P_{\hat{b}^2}) - (P_{\hat{b}} + P_{\hat{b}^3})) \in Z(A) \otimes Z(A) \subset \Delta(A)',$$

where $Z(A)$ is the center of A. In order to show that $\gamma(x) = u\alpha(x)u^*$, $x \in \mathbb{C}(G)$, is an authomorphism of the Kac algebra $\mathcal{A}_\Omega$, we need to verify that $\gamma \circ \kappa_\Omega = \kappa_\Omega \circ \gamma$. Indeed, it is obvious that $\kappa(u) = u^*$. Then $\kappa_\Omega(\gamma(x)) = u^2\alpha(\kappa(x))(u^*)^2$ and $\gamma(\kappa_\Omega(x)) = \alpha(\kappa(x))$ for all $x \in A$. But

$$u^2 = P_{\hat{e}} + -P_{\hat{b}} + P_{\hat{b}^2} - (P_{\hat{b}^3} = \lambda(b^2) \in Z(A).$$

Hence, γ is an authomorphism of the Kac algebra $\mathcal{A}_\Omega$ of order 2.

Let us consider the group $\Gamma = \{\mathrm{id}, \gamma\}$ of authomorphisms of the Kac algebra $\mathcal{A}_\Omega$ and define a conditional expectation P on $\mathcal{A}_\Omega$ via the formula $P(x) = \frac{1}{2}(x + \gamma(x))$. Thus we obtain a Delsart hypergroup $(B = P(A), \tilde{\Delta}_\Omega, \varepsilon, \kappa_\Omega, \mu)$. In order to determine the structure of B, note that $\alpha(e_3) = e_4$ and $\alpha(e_{12}^j) = (-1)^j e_{12}^j$ $(j = 1, \ldots, n-1)$ for odd n and $\alpha(e_i) = e_i$ $(i = 1, \ldots, 4)$, $\alpha(e_{12}^j) = (-1)^j e_{12}^j$, $\alpha(e_{21}^j) = (-1)^j e_{21}^j$ $(j = 1, \ldots, n-1)$ for even n. Since, for odd n,

$$u = e_1 + e_2 + i(e_3 - e_4) + \sum_j{}' (e_{21}^j - e_{12}^j) + \sum_j{}'' (e_{11}^j - e_{22}^j)$$

and, for even n,

$$u = e_1 + e_2 + e_3 + e_4 + \sum_j{}' (e_{21}^j - e_{12}^j) + \sum_j{}'' (e_{11}^j - e_{22}^j),$$

we obtain an explicit formula for the action of γ. Indeed, for odd n, $\gamma(e_3) = e_4$, for even j, we have $\gamma(e_{kl}^j) = e_{kl}^j$, $k, l = 1, 2$, and for odd j, we have $\gamma(e_{11}^j) = e_{22}^j$, $\gamma(e_{12}^j) = e_{21}^j$. For even n, we have $\gamma(e_k) = e_k$, $k = 1, \ldots, 4$, $\gamma(e_{kl}^j) = e_{kl}^j$, $k, l = 1, 2$, for even j, and $\gamma(e_{11}^j) = e_{22}^j$, $\gamma(e_{12}^j) = e_{21}^j$ for odd j. Hence for odd n, we have

$$B_n = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n+2} \oplus \underbrace{M_2(\mathbb{C}) \oplus \cdots \oplus M_2(\mathbb{C})}_{\frac{n-1}{2}}$$

and, for even n,

$$B_n = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n+4} \oplus \underbrace{M_2(\mathbb{C}) \oplus \cdots \oplus M_2(\mathbb{C})}_{\frac{n-2}{2}}.$$

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In any case, $\dim B = 3n$.

Let us show that the comultiplication $\tilde{\Delta}_\Omega$ is not symmetric. For this, it is enough to prove that

$$(P \otimes P)(\Delta_\Omega)(\lambda(ab)) - \Sigma\Delta_\Omega(\lambda(ab))) \neq 0, \qquad (4.1)$$

where Σ is the flip in $A \otimes A$. It is useful to note that $\Omega = \Omega_1 + i\Omega_2$, where $\Sigma\Omega_1 = \Omega_1^* = \Omega_1$ and $\Sigma\Omega_2 = -\Omega_2^* = -\Omega_2$. Then relation (4.1) is equivalent to

$$(P \otimes P)(\Omega_1(\lambda(ab) \otimes \lambda(ab))\Omega_2 - \Omega_2(\lambda(ab) \otimes \lambda(ab))\Omega_1) \neq 0.$$

By direct calculations one can obtain that

$$(P \otimes P)(\Omega_1(\lambda(ab) \otimes \lambda(ab))\Omega_2 - \Omega_2(\lambda(ab) \otimes \lambda(ab))\Omega_1)$$
$$= \sum_k{}' \cos\frac{\pi k}{n}(e_{11}^k + e_{22}^k) \otimes \sum_l{}'' \sin\frac{\pi l}{n}(e_{12}^l - e_{21}^l) \neq 0$$

for even n and

$$(P \otimes P)(\Omega_1(\lambda(ab) \otimes \lambda(ab))\Omega_2 - \Omega_2(\lambda(ab) \otimes \lambda(ab))\Omega_1)$$
$$= \sum_k{}' \cos\frac{\pi k}{n}(e_{11}^k + e_{22}^k) \otimes \sum_l{}'' \sin\frac{\pi l}{n}(-e_{11}^l + e_{22}^l) \neq 0$$

for odd n, where Σ' (resp. Σ'') means that the corresponding index in the summation takes only odd (resp. even) values.

4.2. Remark. The Kac algebra $\mathcal{A}_\Omega$ for $n = 2$ is nothing else but the historical Kac-Paljutkin example of a non-trivial Kac algebra [8]. In this case, the algebra B_2 is commutative but the comultiplication $\tilde{\Delta}_\Omega$ is not symmetric. So B_2 is the usual noncommutative hypergroup of order 6. For $n \geq 3$, the quantum hypergroups $\mathcal{B}_n$ are nontrivial. We obtain exact formulas for the comultiplication in
$$B_3 = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{5} \oplus M_2(\mathbb{C}).$$

Since Q_3 is isomorphic to $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$, one can use the left regular representation of $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ instead of Q_3 (this means that the basis of matrix units of the algebra A is changed up to corresponding authomorphism which does not permit two-dimensional minimal ideals). The formulas for comultiplication in the new basis become simplier. So, in the basis of B_3: $f_1 = P(e_1) = e_1$, $f_2 = P(e_2) = e_2$, $2f_3 = P(e_3) = 1/2(e_3 + e_4) = P(e_4)$, $2f_4 = P(e_{11}^2) = 1/2(e_{11}^2 + e_{22}^2) = P(e_{22}^2)$, $2f_5 = P(e_{12}^2) = 1/2(e_{12}^2 + e_{21}^2) = P(e_{21}^2)$, $f_{ij} = P(e_{ij}^1) = e_{ij}^1$, $i, j = 1, 2$,

we have:

$$
\begin{aligned}
\tilde{\Delta}_\Omega(f_1) &= f_1 \otimes f_1 + f_2 \otimes f_2 + 1/2 f_3 \otimes f_3 + 1/4 f_4 \otimes f_4 + 1/4 f_5 \otimes f_5 \\
&+ 1/2 \left(f_{11} \otimes f_{22} + f_{12} \otimes f_{21} + f_{21} \otimes f_{12} + f_{22} \otimes f_{11} \right), \\
\tilde{\Delta}_\Omega(f_2) &= f_1 \otimes f_2 + 1/2 f_3 \otimes f_3 + f_2 \otimes f_1 + 1/4 f_4 \otimes f_4 + 1/4 f_5 \otimes f_5 \\
&+ 1/2 \left(f_{11} \otimes f_{22} - f_{12} \otimes f_{21} - f_{21} \otimes f_{12} + f_{22} \otimes f_{11} \right), \\
\tilde{\Delta}_\Omega(f_3) &= (f_1 + f_2) \otimes f_3 + f_3 \otimes (f_1 + f_2) + 1/2\{(f_{11} + f_{22}) \otimes f_4 \\
&+ f_4 \otimes (f_{11} + f_{22}) + (f_{11} - f_{22}) \otimes f_5 - f_5 \otimes (f_{11} - f_{22}), \\
\tilde{\Delta}_\Omega(f_4) &= (f_1 + f_2) \otimes f_4 + f_4 \otimes (f_1 + f_2) + (f_{11} + f_{22}) \otimes f_3 + f_3 \otimes (f_{11} + f_{22}) \\
&+ 1/2\{(f_{11} + f_{22}) \otimes f_4 + f_4 \otimes (f_{11} + f_{22}) - (f_{11} - f_{22}) \otimes f_5 \\
&+ f_5 \otimes (f_{11} - f_{22})\}, \\
\tilde{\Delta}_\Omega(f_5) &= (f_1 + f_2) \otimes f_5 + f_5 \otimes (f_1 + f_2) - (f_{11} - f_{22}) \otimes f_3 + f_3 \otimes (f_{11} - f_{22}) \\
&+ 1/2\{(f_{11} - f_{22}) \otimes f_4 - f_4 \otimes (f_{11} - f_{22}) - (f_{11} + f_{22}) \otimes f_5 \\
&- f_5 \otimes (f_{11} + f_{22})\}, \\
\tilde{\Delta}_\Omega(f_{11}) &= f_1 \otimes f_{11} + f_{11} \otimes f_1 + f_2 \otimes f_{11} + f_{11} \otimes f_2 + 1/2 f_3 \otimes (f_4 + f_5) \\
&+ 1/2 (f_4 - f_5) \otimes f_3 + f_{22} \otimes f_{22} + 1/4(f_4 + f_5) \otimes (f_4 - f_5), \\
\tilde{\Delta}_\Omega(f_{12}) &= f_1 \otimes f_{12} + f_{12} \otimes f_1 - f_2 \otimes f_{12} - f_{12} \otimes f_2 + f_{21} \otimes f_{21}, \\
\tilde{\Delta}_\Omega(f_{21}) &= f_1 \otimes f_{21} + f_{21} \otimes f_1 - f_2 \otimes f_{21} - f_{21} \otimes f_2 + f_{12} \otimes f_{12}, \\
\tilde{\Delta}_\Omega(f_{22}) &= f_1 \otimes f_{22} + f_{22} \otimes f_1 + f_2 \otimes f_{22} + f_{22} \otimes f_2 + 1/2 f_3 \otimes (f_4 - f_5) \\
&+ 1/2 (f_4 + f_5) \otimes f_3 + f_{11} \otimes f_{11} + 1/4(f_4 - f_5) \otimes (f_4 + f_5).
\end{aligned}
$$

4.3. Quantum hypergroups associated with a twisting of the dihedral group. Let $G = D_{2n} = \mathbb{Z}_{2n} \rtimes_\alpha \mathbb{Z}_2$ ($2 \le n \in \mathbb{N}$) be the dihedral group with the following action of $\mathbb{Z}_2 = \{1, b\}$ on $\mathbb{Z}_{2n} = \{a^k \ (k = 0, 1, ..., 2n - 1)\}$

$$
\alpha_b(a^k) = a^{2n-k} \ (k = 0, 1, ..., 2n - 1).
$$

The group algebra $A = \mathbb{C}(G)$ is isomorphic to

$$
\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \underbrace{M_2(\mathbb{C}) \oplus \cdots \oplus M_2(\mathbb{C})}_{n-1}.
$$

Let e_1, e_2, e_3, e_4, e_{11}^j, e_{12}^j, e_{21}^j, e_{22}^j ($j = 1, ..., n - 1$) be the matrix units of this algebra; we can now write the left regular representation λ of G ([6], 27.61):

$$
\begin{aligned}
\lambda(a^k) &= e_1 + e_2 + (-1)^k(e_3 + e_4) + \sum_j (\varepsilon_n^{jk} e_{11}^j + \varepsilon_n^{jk(2n-1)} e_{22}^j), \\
\lambda(ba^k) &= e_1 - e_2 - (-1)^k(e_3 - e_4) + \sum_j (\varepsilon_n^{jk(2n-1)} e_{12}^j + \varepsilon_n^{jk} e_{21}^j),
\end{aligned}
$$

where $\varepsilon_n = e^{i\pi/n}$.

Consider the Abelian subgroup $H = \{e, a^n, b, ba^n\}$ which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since the dual group $\hat{H}$ is isomorphic to H, one can compute the orthogonal projections $P_{\hat{h}}$, $(\hat{h} \in \hat{H})$ (see [14]):

$$P_{\hat{e}} = e_1 + \frac{1 + (-1)^n}{2} e_4 + q_1,$$

$$P_{\hat{b}} = e_2 + \frac{1 + (-1)^n}{2} e_3 + q_2,$$

$$P_{\hat{a}^n} = \frac{1 - (-1)^n}{2} e_4 + p_1,$$

$$P_{\hat{b}\hat{a}^n} = \frac{1 - (-1)^n}{2} e_3 + p_2,$$

where p_1, p_2, q_1, q_2 are the orthogonal projections defined by :

$$p_1 = 1/2 {\sum_{j}}' (e_{11}^j + e_{12}^j + e_{21}^j + e_{22}^j),$$

$$p_2 = 1/2 {\sum_{j}}' (e_{11}^j - e_{12}^j - e_{21}^j + e_{22}^j),$$

$$q_1 = 1/2 {\sum_{j}}'' (e_{11}^j + e_{12}^j + e_{21}^j + e_{22}^j),$$

$$q_2 = 1/2 {\sum_{j}}'' (e_{11}^j - e_{12}^j - e_{21}^j + e_{22}^j),$$

where $\sum'$ (resp., $\sum''$) means that the corresponding index in the summation takes only odd (resp., even) values. The orthogonal projections $P_{\hat{e}} + P_{\hat{b}}$ and $P_{\hat{a}^n} + P_{\hat{b}\hat{a}^n}$ are central.

Let us consider the 2-cocycle ω on $\hat{H} \times \hat{H}$ such that, for all u, h in $\hat{H}$, $\omega(\hat{e}, u) = \omega(u, u) = 1$, $\overline{\omega(h, u)} = \omega(u, h)$, and $\omega(\hat{a}^n, \hat{b}) = \omega(\hat{b}, \hat{b}\hat{a}^n) = \omega(\hat{b}\hat{a}^n, \hat{a}^n) = i$.

Let Ω be the 2-cocycle of $A \otimes A$ obtained by the lifting construction,

$$
\begin{aligned}
\Omega & = & P_{\hat{e}} \otimes I + P_{\hat{a}^n} \otimes (P_{\hat{e}} + P_{\hat{a}^n} + i(P_{\hat{b}} - P_{\hat{b}\hat{a}^n})) + P_{\hat{b}} \otimes (P_{\hat{e}} + P_{\hat{b}} \\
& + & i(P_{\hat{b}\hat{a}^n} - P_{\hat{a}^n})) + P_{\hat{b}\hat{a}^n} \otimes (P_{\hat{e}} + P_{\hat{b}\hat{a}^n} + i(P_{\hat{a}^n} - P_{\hat{b}})).
\end{aligned}
$$

We can also write $\Omega = \Omega_1 + i\Omega_2$, where $\Sigma\Omega_1 = \Omega_1 = \Omega_1{}^*$, and $\Sigma\Omega_2 = -\Omega_2 = -\Omega_2{}^*$, where Σ is the flip in $A \otimes A$. It is clear that the 2-cocycle Ω is strongly co-involutive on (A, Δ, κ). So $\mathcal{A}_\Omega = (A, \Delta_\Omega, \varepsilon, \kappa, \mu)$ is a nontrivial Kac algebra [14].

Let $\gamma(a) = a^p$ be an involute automorphism of $\mathbb{Z}_{2n}$, where $p < 2n - 1$ has no common divisors with $2n$, $n \geq 4$, and $p^2 - 1 = 0$ modulo $2n$. It is clear that we can extend γ to the group G by setting $\gamma(b) = b$. Since γ acts trivially on H, we have, by Proposition 3.6, that γ is an automorphism of the Kac algebra $\mathcal{A}_\Omega$. Let us consider the group $\Gamma = \{\text{id}, \gamma\}$ of authomorphisms of the Kac algebra $\mathcal{A}_\Omega$ and

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define a conditional expectation P on $\mathcal{A}_\Omega$ via the formula $P(x) = \frac{1}{2}(x + \gamma(x))$. Thus we obtain a quantum Delsart hypergroup $(B = P(A), \tilde{\Delta}_\Omega, \varepsilon, \kappa, \mu)$. Since $p \neq 2n-1$, we have that B is noncommutative (note that, if $\gamma^2 \neq \mathrm{id}$, this is not true in general). The dimension of B equals the number of Γ-orbits, so we have $\dim B = 2n + r$, where r is the number of solutions of the equation $x(p-1) = 0$ modulo $2n$. If n is a prime number, then $\dim B = 2n + 2$.

For example, if $n = 4$ and $\gamma(a) = a^3$, then we have $\gamma(e_i) = e_i$, $i = 1, \ldots, 4$, $\gamma(e_{ij}^1) = e_{ij}^3$, $i, j = 1, 2$, and $\gamma(e_{11}^2) = e_{22}^2$, $\gamma(e_{12}^2) = e_{21}^2$. Thus

$$B = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{6} \oplus M_2(\mathbb{C}).$$

Let us show that the comultiplication $\tilde{\Delta}_\Omega$ is not symmetric. For this it is enough to prove that

$$(P \otimes P)(\Delta_\Omega)(\lambda(a)) - \Sigma\Delta_\Omega(\lambda(a))) \neq 0,$$

where Σ is the flip in $A \otimes A$, or, equivalently, that

$$(P \otimes P)(\Omega_1(\lambda(a) \otimes \lambda(a))\Omega_2 - \Omega_2(\lambda(a) \otimes \lambda(a))\Omega_1) \neq 0.$$

Since

$$\begin{aligned}
\Omega_1 &= \frac{1}{2}\lambda(e) \otimes \lambda(e) + \frac{1}{8}\{\lambda(e) \otimes (\lambda(e) + \lambda(a^n) + \lambda(b) + \lambda(ba^n)) \\
&+ \lambda(a^n) \otimes (\lambda(e) + \lambda(a^n) - \lambda(b) - \lambda(ba^n)) \\
&+ \lambda(b) \otimes (\lambda(e) - \lambda(a^n) + \lambda(b) - \lambda(ba^n)) \\
&+ \lambda(ba^n) \otimes (\lambda(e) - \lambda(a^n) - \lambda(b) + \lambda(ba^n))\}
\end{aligned}$$

and

$$\begin{aligned}
\Omega_2 &= \frac{1}{4}\{\lambda(a^n) \otimes \lambda(b) - \lambda(a^n) \otimes \lambda(ba^n) - \lambda(b) \otimes \lambda(a^n) + \lambda(b) \otimes \lambda(ba^n) \\
&+ \lambda(ba^n) \otimes \lambda(a^n) - \lambda(ba^n) \otimes \lambda(b)\},
\end{aligned}$$

we have

$$\begin{aligned}
\Omega_2\lambda(a) &\otimes \lambda(a)\Omega_1 - \Omega_1\lambda(a) \otimes \lambda(a)\Omega_2 \\
&= \frac{1}{4}\lambda(a^{n-1}) \otimes (\lambda(a^{2n-1}) - \lambda(a^{n-1}) - \lambda(ba^{n+1}) + \lambda(ba)) \\
&+ \frac{1}{2}(\lambda(a^{n+1}) - \lambda(a^{2n-1})) \otimes \lambda(b)[(\lambda(a) - \lambda(a^{n+1})) + (\lambda(a^{n-1}) - \lambda(a^{2n-1}))] \\
&+ \frac{1}{2}\lambda(b)(\lambda(a^{n+1}) + \lambda(a^{2n-1})) \otimes (\lambda(a^{n-1}) - a^{2n-1}) \\
&+ \frac{1}{2}\lambda(b)(\lambda(a) - \lambda(a^{n-1})) \otimes \lambda(b)(\lambda(a^{n+1}) - \lambda(a^{2n-1})) \\
&+ \frac{1}{2}\lambda(b)(\lambda(a) + \lambda(a^{n-1})) \otimes (\lambda(a^{2n-1}) - \lambda(a^{n+1})) \\
&+ \frac{1}{2}\lambda(b)(\lambda(a^{2n-1}) - \lambda(a^{n+1})) \otimes \lambda(b)(\lambda(a) - \lambda(a^{n-1})).
\end{aligned}$$

Apply $P \otimes P$ to the both sides of this equality. Then the sum of the first three summands is equal to zero iff $p = n + 1$, while the sum of the second three summands is equal to zero iff $p = n - 1$. Since $n \geq 4$, we obtain that the comultiplication $\tilde{\Delta}_\Omega$ is not symmetric.

4.4. Quantum hypergroups associated with a twisting of the symmetric group. The twisting of the symmetric group S_n, $n \geq 4$, by a 2-cocycle lifted from the Abelian subgroup $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the permutations $a = (12)$ and $b = (34)$ was constructed in [10]. The 2-cocycle ω on $\hat{H} \times \hat{H}$ is defined by $\omega(\hat{a}, \hat{b}) = \omega(\hat{b}, \hat{a}\hat{b}) = \omega(\hat{a}\hat{b}, a) = i$, $\omega(\hat{e}, \hat{x}) = \omega(\hat{x}, \hat{e}) = \omega(\hat{x}, \hat{x}) = 1$, $\omega(\hat{x}, \hat{y}) = \overline{\omega(\hat{y}, \hat{x})}$ for all $\hat{x}, \hat{y} \in \hat{H}$. Let Ω be the lifted counital 2-cocycle, as in 3.4. Then the twisted Kac algebra $\mathcal{A}_\Omega$ is non-symmetric.

Denote by γ the inner automorphism of the group S_n generating by the permutation a. Since γ acts trivially on the subgroup H, by Proposition 3.6 we have that γ is an automorphism of the twisted Kac algebra $\mathcal{A}_\Omega$ of order 2. Let P be the conditional expectation on $\mathcal{A}_\Omega$ associated with the subgroup $\Gamma = \{\mathrm{id}, \gamma\}$. Then $B = P(\mathbb{C}(S_n))$ is a quantum hypergroup. The algebra B is noncommutative, since $\lambda(b)$ does not commute with $P(\lambda(c))$, where $c = (2341)$. The dimension of B equals the number of Γ-orbits, so we have $\dim B = \frac{1}{2}(n^2 - n + 2)(n - 2)!$.

Let us show that the comuliplication $\tilde{\Delta}_\Omega$ is not symmetric. We can write $\Omega = \Omega_1 + i\Omega_2$, where $\Sigma\Omega_1 = \Omega_1 = \Omega_1{}^*$ and $\Sigma\Omega_2 = -\Omega_2 = -\Omega_2{}^*$, where Σ is the flip in $A \otimes A$. Thus, as in the previous example, our statement follows from the inequality

$$(P \otimes P)(\Omega_1(\lambda(c) \otimes \lambda(c))\Omega_2 - \Omega_2(\lambda(c) \otimes \lambda(c))\Omega_1) \neq 0 \tag{4.2}$$

which can be obtained by strightforward calculations with $c = (2341)$.

4.5. Quantum hypergroups associated with a twisting of the alternating group. A nontrivial twisting of the alternating group A_n, $n \geq 4$, by a 2-cocycle lifted from the abelian subgroup $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the elements $a = (12)(34)$ and $b = (13)(24)$ was constructed in [10]. The 2-cocycle ω on $\hat{H} \times \hat{H}$ is the same as in 4.5. Let Ω be the lifted counital 2-cocycle, as in 3.4. Then the twisted Kac algebra $\mathcal{A}_\Omega$ is non-symmetric iff $n \geq 5$.

Let γ be the restriction of the inner automorphism of the group S_n generated by the permutation (12) to the group A_n. It is easy to verify that relation (3.5) holds for the choosen automorphism γ. Hence, γ is an automorphism of the twisted Kac algebra $\mathcal{A}_\Omega$ by virtue of Remark 3.8. Let P be the conditional expectation on $\mathcal{A}_\Omega$ associated with the subgroup $\Gamma = \{\mathrm{id}, \gamma\}$. Then $B = P(\mathbb{C}(A_n))$ is the quantum hypergroup. It is obvious that the algebra B is noncommutative and $\dim B = \frac{1}{4}(n^2 - n + 2)(n - 2)!$.

We can obtain, by strightforward calculations, that ineqiality (4.2) holds with $c = (345) \in A_n$, $n \geq 5$. Thus the comuliplication $\tilde{\Delta}_\Omega$ is not symmetric.

4.4. Quantum hypergroups associated with a twisting of the group $\mathbb{Z}_m^2 \rtimes \mathbb{Z}_2$. Let $G = \mathbb{Z}_m^2 \rtimes \mathbb{Z}_2$, $m \geq 3$, be a finite group of order $2m^2$ with the

following action α of $\mathbb{Z}_2 = \{\mathrm{id}, s\}$ on $H = \mathbb{Z}_m^2 = \{(a,b)|a,b = 0, 1, \ldots, m-1\}$: $\alpha_s(a,b) = (b,a)$. The twisting of the group G was constructed in [14] by using the 2-cocycle ω on $\hat{H} = \hat{\mathbb{Z}}_m^2 \times \hat{\mathbb{Z}}_m^2$:

$$\omega(\hat{a}, \hat{b}; \hat{c}, \hat{d}) = \exp\left(\frac{2\pi i}{m}(\hat{a}\hat{d} - \hat{b}\hat{c})\right).$$

Let Ω be the counital 2-cocycle on $\mathbb{C}(G) \otimes \mathbb{C}(G)$ obtained by the lifting construction:

$$\Omega = \sum_{\hat{H} \times \hat{H}} \exp\left(\frac{2\pi i}{m}(\hat{a}\hat{d} - \hat{b}\hat{c})\right) P_{(\hat{a}, \hat{b})} \otimes P_{(\hat{c}, \hat{d})}.$$

Then the twisted Kac algebra $\mathcal{A}_\Omega$ is not symmetric, for example, $\Delta_\Omega(\lambda(s)) \neq \Sigma\Delta_\Omega(\lambda(s))$.

Define an automorphism γ of the group G as follows: $\gamma(s) = s$ and $\gamma(a,b) = (ar, br)$, where $(a,b) \in H$ and $r^2 = 1$ modulo m. Then it is strightforward that $(\gamma \otimes \gamma)(\Omega) = \Omega$. Thus γ is an automorphism of the twisted Kac algebra $\mathcal{A}_\Omega$ by Proposition 3.6. Let P be the conditional expectation on $\mathcal{A}_\Omega$ associated with the subgroup $\Gamma = \{\mathrm{id}, \gamma\}$. Then $B = P(\mathbb{C}(G))$ is a quantum hypergroup. Since $\gamma(s) = s$, the algebra B is noncommutative. The dimension of B is equal to the number of Γ-orbits, hence $\dim B = m^2 + p^2$, where p is the number of solutions of the equation $(r-1)k = 0$ modulo m.

If $r = m - 1$, then the quantum hypergroup B is non-symmetric. Indeed, it is easy to see that $(\gamma \otimes \mathrm{id})(\Omega) = \Omega^* = (\mathrm{id} \otimes \gamma)(\Omega)$ and $\Sigma\Omega = \Omega^*$. Then, for any $a \in H$, we have

$$
\begin{aligned}
(P \quad \otimes \quad & P)(\Delta_\Omega(\lambda(as)) - \Sigma\Delta_\Omega(\lambda(as))) \\
= \quad & \frac{1}{4}(\mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \gamma + \gamma \otimes \mathrm{id} + \gamma \otimes \gamma)(\Omega\lambda(as) \otimes \lambda(as)\Omega^* - \Omega^*\lambda(as) \otimes \lambda(as)\Omega) \\
= \quad & (\lambda(a) - \lambda(a^{-1})) \otimes (\lambda(a) - \lambda(a^{-1}))(\Omega\lambda(s) \otimes \lambda(s)\Omega^* - \Omega^*\lambda(s) \otimes \lambda(s)\Omega),
\end{aligned}
$$

since Ω commutes with $\mathbb{C}(H)$. Thus $\hat{\Delta}_\Omega$ is non-symmetric iff $(\lambda(a) - \lambda(a^{-1})) \otimes (\lambda(a) - \lambda(a^{-1})) \left(\Omega^2\lambda(s) \otimes \lambda(s) - \lambda(s) \otimes \lambda(s)\Omega^2\right) \neq 0$. This inequality can be obtained by strightforward calculations.

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